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A SURVEY OF INCOMPRESSIBLE, TWO-DIMENSIONAL UNSTEADY BOUNDARY LAYERS

by

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A SURVEY OF INCROMPRESSIBLE, TWO-DIMENSIONAL UNSTEADY BOUNDARY LAYERS.

1.1. Introduction.

Boundary layers are formed on bodies in moving fluids due to the action of viscosity. There are many types of flow and body configurations and thus the range of unsteady boundary layers is large. For the purpose of this paper, only a narrow class of incompressible, two-dimensional boundary layers will be discussed. Thus compressible flow of all types will be excluded as will boundary layers on curved surfaces, rotating surfaces and any other three dimensional bodies.

The term unsteady is further restricted to describe motions in only three forms: impulsive starting, accelerated and harmonic motions. Strictly speaking, the impulsive starting is assumed to be acceleration over a distance short compared with the body length.

Despite these restrictions the paper will bring out the essential features of the subject by setting out the basic equations and then progressively working through the important problems and solutions relevant to the subject.

2.1. The Navier Stokes' Equations.

The subject is of course founded on a study of the Navier Stokes' Equations which in incompressible flow are most simply written as:

$$\rho \frac{Dw}{Dt} = \mu \nabla^2 w - \text{grad.}\, p \qquad \text{Eqn. 1.}$$

where ρ = fluid density
w = velocity vector
μ = viscosity
p = surface force.

The equation of continuity is often required and is:

$$\text{div}\, w = 0 \qquad \text{Eqn. 2.}$$

If the flow is assumed independant of the z direction equation 1 becomes:

$$\rho \frac{\partial u}{\partial t} = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2} \qquad \text{Eqn. 3.}$$

For cases where boundary layer theory is applicable, the pressure gradient is:

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{\partial U}{\partial t} - U\frac{\partial U}{\partial x}$$

3.1. Impulsively Started Plane Wall.

Although this problem is often referred to Rayleigh, Stokes' obtained the solution first and thus it is more correctly known as Stokes' first problem. The nature of a true impulsive start is probably very difficult to define and although the problem refers to it as such, the real implied meaning is "acceleration over a distance very short in comparison with body length scales".

Nevertheless, a solution can be found to the problem of a semi infinite region of incompressible fluid bounded by an impervious rigid plane undergoing a sudden acceleration to a steady velocity.

For zero pressure gradient equation 3 becomes:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{Eqn. 4.}$$

which is also known as a diffusion equation. The boundary conditions to be satisfied are:

$$t \le 0 \;; \quad u = 0$$

$$t > 0 \;; \quad u = U_0 \text{ at } y = 0$$

$$u = 0 \text{ at } y = \infty$$

The substitution

$$\eta = y\left(2\sqrt{\nu t}\right)^{-1}$$

and assumption

$$u = U_0 f(\eta)$$

enables the ordinary differential equation to be obtained:

$$f'' + 2\eta f' = 0$$

with boundary conditions

$$f = 1 \quad \text{at} \quad \eta = 0$$

$$f = 0 \quad \text{at} \quad \eta = \infty$$

The solution is:

$$u = U_0 \operatorname{erfc} \eta \qquad \text{Eqn. 5.}$$

where $$\operatorname{erfc} = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} \exp - \eta^2 d\eta$$

The velocity distribution resulting from equation 5 is shown in Fig. 1.

Boundary Layer Thickness.

Defining the boundary layer depth as that at which the velocity is 99% of the free stream value, it is clear that when $\operatorname{erfc} = 0.01$ this results in the equation giving the boundary layer depth. At $\operatorname{erfc} = 0.01$, η is given as 2.0. Thus

$$\delta \simeq 4\sqrt{\nu t} \qquad \text{Eqn. 6.}$$

This represents the depth of penetration of vorticity created at the plate and its outward diffusion. It has a similar analogy in heat conduction.

A more complete Solution.

The result given by equation 5 is the first approximation for both two dimensional and axisymetric cases. However it is not the complete solution to the problem of an impulsively started plane wall. If the wall is allowed to be of finite length upstream then a further condition must be satisfied:

at x = 0, u = 0 for all y, t

Effectively then, equation 5 is the solution when the upstream edge is sufficiently far away to have negligible effect on the flow.

Blasius considered this problem and assumed that a stream function could be defined as a power series in time:

$$\psi(x,y,t) = 2\sqrt{\nu t}\left\{U \zeta_0(\eta) + t\, U \frac{dU}{dx} \zeta_1(\eta) + \cdots\right\} \qquad \text{Eqn. 7.}$$

where ζ_0 and ζ_1 are functions of η

as $$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x}$$

$$u = U\zeta_0' + tU\frac{dU}{dx}\zeta_1' + \dots$$

$$v = -2\sqrt{\nu t}\left\{\frac{dU}{dx}\zeta_0 + t\left[\left(\frac{dU}{dx}\right)^2 + U\frac{d^2U}{dx^2}\right]\zeta_1 + \right\}$$

Now in the first instance after the motion has started from rest the boundary layer is very thin and the viscous term is very large whereas the convective terms are of normal size.

The viscous term is then balanced by the non steady acceleration $\frac{\partial u}{\partial t}$ together with the pressure term of which the contribution $\frac{\partial U}{\partial t}$ is of most importance.

If we allow the velocity to consist of two terms

$$u(x, y, t) = u_0(x, y, t) + u_1(x, y, t)$$

The first approximation u_0 thus satisfies the equation

$$\frac{\partial u_0}{\partial t} = \frac{\partial U}{\partial t} + \nu\frac{\partial^2 u_0}{\partial y^2} \qquad \text{Eqn. 8.}$$

with boundary conditions $\quad y = 0, \; u_0 = 0$
$\quad y = \infty \quad u_0 = U(x, t)$

(assuming a system of coordinates stationary relative to the plane wall).

Putting in u and v into equation 8 results in

$$\zeta_0''' + 2\eta\zeta_0'' = 0$$

as a first approximation, with boundary conditions:

$$\zeta_0 = \zeta_0' = 0 \quad \text{at} \quad \eta = 0.$$
$$= 1 \quad \text{at} \quad \eta = \infty$$

Equation 8 results in the solution given in equation 5 and Fig. 1.

Returning to the approximation culminating in equation 8, the equation for the second approximation u_1 is obtained by calculating the convective terms from u_0. This results in:

$$\frac{\partial u_1}{\partial t} - \nu\frac{\partial^2 u_1}{\partial y^2} = U\frac{dU}{\partial x} - u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y} \qquad \text{Eqn. 9.}$$

with boundary conditions:

$$u_1 = 0 \quad \text{at} \quad y = 0$$
$$u_1 = 0 \quad \text{at} \quad y = \infty$$

Thus a differential equation in ζ_1 results in:

$$\zeta_1''' + 2\eta\zeta_1'' - 4\zeta_1' = 4\left(\zeta_0'^2 - \zeta_0\zeta'' - 1\right)$$

with boundary conditions

$$\zeta_1 = \zeta_1' = 0 \quad \text{at } \eta = 0$$
$$\zeta_1' = 0 \quad \text{at } \eta = \infty$$

The solution for the second approximation as derived by Blasius is

$$\zeta_1 = -\frac{3}{\sqrt{\pi}}\eta \exp -\eta^2 \operatorname{erfc}\eta + \frac{1}{2}(2\eta^2 - 1)\operatorname{erfc}^2\eta + \frac{2}{\pi}\exp -2\eta^2 + \frac{1}{\sqrt{\pi}}\eta\exp -\eta^2$$
$$+ 2\operatorname{erfc}\eta - \frac{4}{3\pi}\exp -\eta^2 + \left(\frac{2}{\sqrt{\pi}} + \frac{4}{3\pi^{3/2}}\right)\left\{\eta \exp -\eta^2 - \frac{\sqrt{\pi}}{2}(2\eta^2+1)\operatorname{erfc}\eta\right\}$$

<u>Transition from Stokes' first problem flow to Blasius impulsively started plane flow.</u>

Stewartson (1951) considered in more detail the connection between an infinite and finite plate moving from rest impulsively. The first method of approximation used Rayleigh's analogy (Rayleigh also solved Stokes' first problem) in which it is assumed that all convection takes place at the free stream velocity.

Thus the term $U\frac{\partial U}{\partial x}$ is dominant in equation 1

i.e. $$\frac{\partial u}{\partial t} = -U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{Eqn. 11.}$$

by defining $$\zeta = \frac{y}{\sqrt{\nu t}}$$
$$\tau = \frac{Ut}{x}$$

equation 11 can be shown to have the solutions

$$u = U \operatorname{erfc} \zeta/2 \qquad \text{when} \qquad \tau < 1.$$
$$u = U \operatorname{erfc}\left(\zeta t^{1/2}/2\right) \qquad \text{when} \qquad \tau > 1$$

Thus for $\tau < 0$ the flow is independent of x (Blasius flow) but for $\tau > 1$ the flow is of Rayleigh type. This means that as the time for motion increases, the flow over a plate changes at $\tau = 1$ from Blasius to Rayleigh flow. This first approximation is valid only near the edge of the boundary layer.

Higher order approximations still show the existance of mathematical singularities or an abrupt change in the flow characteristics - not a likely physical reality.

3.2. External flow undergoing an impulsive increase in velocity.

Watson (1958) considered the problem of the external flow velocity over a plate undergoing an impulsive increase to a new value. Thus $U(t)$ is constant at one value for $t < 0$ and a higher constant at $t > 0$. His analysis showed that the skin friction rises sharply at the moment of impulse and then gradually decays to a higher limiting value. This results from the formation of a secondary boundary layer next to the plate and whose thickness is of the order $\sqrt{\nu t}$. This secondary layer is described by Stokes' first problem (or Rayleigh layer) in the beginning and its growth gradually causes the distortion of the primary layer to its new shape.

A point of considerable interest arising from this feature is that a separating boundary layer when exposed to a sharp increase in free stream velocity must experience an increase in skin friction such that the point of separation is moved further downstream. This may account for stall hysterisis effects to be described in more detail later on.

3.3. Boundary layer formation in accelerated motion.

Blasius also calculated the boundary layer formation for accelerated motion of a body and the results are very similar to those of impulsive starting.

The potential velocity of the body is now given as:

$$t \leq 0 \qquad u(x, t) = 0$$
$$t > 0 \qquad u(x, t) = t.w(x)$$

Following the method of successive approximations outlined earlier, a stream function is defined as

$$\psi(x, y, t) = 2\sqrt{\nu t}\left\{t\,w\,\xi_0(\eta) + t^3 w \frac{dw}{dx}\xi_1\eta + \cdots\right\}$$

$$u(x, y, t) = u\left(\xi_0' + t^2 \frac{dw}{dx}\xi_1' + \cdots\right)$$

and equations for ξ_0 and ξ_1 result in:

$$\xi_0''' + 2\eta\xi_0'' - 4\xi_0' = -4$$
$$\xi_1''' + 2\eta\xi_1'' - 12\xi_1' = -4 + 4(\xi_0'' - \xi_0\xi_0'')$$

with boundary conditions

$$\eta = 0 \qquad \xi_0 = \xi_0' = 0, \quad \xi_1 = \xi_1' = 0$$

$$\eta = \infty \qquad \xi_0' = 1, \quad \xi_1' = 0$$

The solution for ζ_0' is given as

$$\zeta_0' = 1 + \frac{2}{\sqrt{\pi}}\eta \exp -\eta^2 (1 + 2\eta^2) \operatorname{erf} \eta \qquad \text{Eqn. 12.}$$

3.4. The flow near an oscillating flat plate.

This problem was first solved by Stokes and is known as Stokes' second problem. This starts the discussion of harmonic motions of either the body or the free stream. It is a fact that steady flow is a limiting case of unsteady flow and this serves to emphasize the importance of its study.

Stokes' second problem was concerned with a time averaged stationary semi infinite volume of fluid bounded by an infinite impervious rigid plate performing harmonic oscillations.

Thus the boundary conditions defining the problem are:

$$y = 0, \quad u = U_0 \cos nt \quad \text{for all } t$$
$$y = \infty, \quad u = 0$$

defining terms $k = \sqrt{\frac{n}{2\nu}}$ and $\eta = y\sqrt{\frac{n}{2\nu}}$
equation 4 can be solved assuming a separable solution to give:

$$u(y,t) = U_0 e^{-\eta} \cos(nt - \eta) \qquad \text{Eqn. 13.}$$

where n = harmonic oscillation frequency in rad/sec.

The solution is sketched in Fig.2 for several instants of time. The interesting feature of this damped harmonic wave is that fluid layers $2\pi\sqrt{2\nu/n}$ apart oscillate in phase and this separation is sometimes known as a depth of penetration of the viscous wave. The boundary layer depth has a thickness of order

$$\delta \simeq \sqrt{\nu/n} \qquad \text{Eqn. 14.}$$

This also has a similar analogy in heat transfer. By an appropriate coordinate transformation, the situation of a stationary surface under an oscillating free stream can be obtained. This necessarily involves a virtual pressure field to be introduced to account for inertial effect. Alternatively, the periodic solution of equation 3 with

$$-nU_0 \sin nt = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

with boundary conditions $u = 0 \quad y = 0$

$u = U_0 \cos nt \quad y = \infty$

is $$u = U_0 \cos nt - U e^{-\eta} \cos(nt - \eta) \qquad \text{Eqn. 15.}$$

Thus an equal and opposite oscillating velocity has merely been added to equation 5.

The principle feature of interest contained in equation 15 is that the skin friction or velocity gradient at the surface leads the free stream velocity by $\pi/4$

3.5. Unsteady Couette flow.

This problem is similar to Stokes' second problem but with the flow fluctuating velocity not zero at infinity but a distance δ from the plate. Thus we can provide a solution for two out of phase oscillating infinite parallel plate. (Couette flow). The problem again has a similar analogy in heat transfer, and the solution as found by the author is given below.

Equation 4 is therefore to be solved with the boundary conditions

$$u(0, t) = 0 \quad \text{at} \quad y = 0$$
$$u(\delta, t) = U_0 \cos nt \quad \text{at} \quad y = \delta$$

where δ = distance apart of plates

n = harmonic frequency

Assuming a separable solution and writing

$$b = \sqrt{\frac{n y^2}{2\nu}}$$
$$c = nt$$
$$d = \sqrt{\frac{n \delta^2}{2\nu}}$$

the general solution becomes

$$u = \frac{U_0}{2}\left[\frac{\cos d \sinh d\left(e^{b}\cos(b+c) - e^{-b}\cos(c-b)\right) + \sin d \cosh d\left(e^{b}\sin(b+c) - e^{-b}\sin(c-b)\right)}{\cos^2 d \sinh^2 d + \sin^2 d \cosh^2 d}\right] \qquad \text{Eqn. 16.}$$

This can be considerably simplified when b and d are much smaller than unity. Hence

$$u \simeq U_0 \frac{b}{d} \cos nt$$

$$u \simeq U_0 \frac{y}{\delta^2} \cos nt \qquad \text{Eqn. 17.}$$

Also the shear stress is proportional to $\frac{du}{dy}$.

$$\frac{du}{dy} = -U_0 \frac{2y}{\delta^2} nt \sin nt$$

so at the wall the shear stress becomes zero.

Another simplification can be made when b and d are large - typically above π

Hence
$$u \simeq U_0 \frac{e^b}{e^d} \cos(d - b - c)$$

$$u \simeq U_0 \frac{e^{\sqrt{\frac{ny^2}{2\nu}}}}{e^{\sqrt{\frac{n\delta^2}{2\nu}}}} \cos\left(\sqrt{\frac{n\delta^2}{2\nu}} - \sqrt{\frac{ny^2}{2\nu}} - nt\right)$$

Plots of varying instants in time and plate separations can be seen in Fig. 3.

3.6. Periodic Boundary Layer in the absence of a mean flow.

For the cases where boundary layer theory is applicable equations 1 and 2 written more fully for two dimensional flow are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

and
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

When considering the case of an oscillating free stream above a plate, Schlichting (1932) followed the approach outlined below:

Rewriting the equation above:

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial U}{\partial t} = U\frac{\partial U}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

if $\frac{\nu}{nl^2}$ is small, (where l = typical length scale parallel to the wall) the terms on the right hand side are zero or small compared to those on the left. Providing we intend to find the first order oscillating motion and the dominant part of the resulting second order steady motion then the boundary layer approximated equations above are valid.

Defining $\eta = y\sqrt{\frac{n}{2\nu}}$

a first approximation to the stream function φ_1 is

$$\varphi_1 = \left(\frac{2\nu}{n}\right)^{1/2} U_0(x)\,\zeta_1(\eta)\,e^{int} \qquad \text{Eqn. 18.}$$

where the free stream oscillating velocity is

$$U = U_0(x)\cos nt$$

ζ_1 satisfies $\zeta_1''' + 2i\,(1+\zeta_1') = 0$

with conditions $\zeta_1 = \zeta_1' = 0$ at $\eta = 0$

$\zeta' \rightarrow 1$ as $\eta \rightarrow \infty$

thus $\zeta_1 = -\frac{(1-i)}{2}\left(1 - e^{-(1+i)\eta}\right) + \eta$

This refers to that part of the boundary layer which has a harmonic response similar in frequency to the external flow. The skin friction has a phase lead of $\pi/4$ over the velocity fluctuations.

The component of velocity parallel to the wall is given by equation 15 but the flow velocity normal to the wall is:

$$V_1 = -\left(\frac{2\nu}{n}\right)^{1/2} U_0'(x)\left\{\eta\cos nt + \frac{1}{\sqrt{2}}\cos\left(nt + \frac{7}{4}\pi\right) + \frac{e^{-\eta}}{\sqrt{2}}\cos\left(nt - \eta - \frac{\pi}{4}\right)\right\} \qquad \text{Eqn. 19.}$$

Outside the boundary layer the third term tends to zero; the first term is the contribution resulting from continuity and the second term, the displacement effect of the boundary layer on the external flow, otherwise known as the diffusion of periodic vorticity.

A higher order approximation predicts a second harmonic response or non linearity and also that a steady flow exists outside the boundary layer. The steady motion is generated by the Reynolds' stress associated with the osillatory part of the flow within the boundary layer and its persistance outside the oscillatory boundary layer because of the action of vorticity.

A Reynolds' number based on the steady velocity of the streaming motion and a typical body length was found to be an important parameter in the va. lity of the equations.

If Re is small enough for linearisation of the Navier Stokes' equations to be valid, Rayleigh showed that periodic vortices are formed above the surface. This was an explanation of the movement of dust particles in a Kundt tube.

If Re is large, there exists a second outer boundary layer at the edge of which the velocity is zero. Within this layer it is not valid to neglect the non linear inertia terms and the usual boundary layer ideas show that its thickness is of the order

$$t = \frac{l\sqrt{\nu n}}{U_\infty}$$

where l = body length

U_∞ = body maximum velocity

This is much thicker than the inner boundary layer of thickness $\simeq \sqrt{\nu/n}$ because $\frac{nl}{U_\infty}$ was assumed large.

3.7. A laminar boundary layer with fluctuations imposed on the free stream velocity.

This problem was solved by both Lin (1956) and Lighthill (1954) and only considers fluctuations in magnitude and not direction of the free stream.

A frequency parameter of im portance is $\frac{n\delta^2}{\nu}$ where δ is the unperturbed boundary layer depth. It is proportional to $\frac{nU_\infty}{l}$ where U_∞ is the mean free stream velocity and l a typical length parallel to the body. The free stream velocity consists of a steady value U_∞ and a small harmonic perturbation $\epsilon U_\infty e^{int}$ At the edge of the boundary layer,

$$U(x,t) = U_0(x)(1 + \epsilon e^{int})$$

Allowing the velocities u and v in the boundary layer to have harmonic perturbations gives

$$u = U_0(x,y) + \epsilon u_1 e^{int}$$
$$v = V_0(x,y) + \epsilon V_1(x,y) e^{int}$$

where U_1 and V_1 are the fluctuating parts of the velocities in the boundary layer.

By assuming that ϵ is small linearisation produces for the fluctuating component

$$inu_1 + U_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial U_0}{\partial x} + V_0\frac{\partial u_1}{\partial y} + V_1\frac{\partial u_1}{\partial y} = inU_0 + \frac{d}{dx}(U_0^2) + \nu\frac{\partial^2 u_1}{\partial y^2} \qquad \text{Eqn. 20.}$$

and $\dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} = 0$

Now U_0 is a function of $\frac{x}{l}$ and $y\sqrt{U_\infty/\nu l}$ so as n tends to zero

$$U_s = U_0 + \tfrac{1}{2} y \frac{\partial U_0}{\partial y}$$
$$V_s = \tfrac{1}{2} V_1 \left(y \frac{\partial V_0}{\partial y} \right)$$

Now putting $(U_1, V_1) = (U_s, V_s) + (U_n, V_n)$ where (U_n, V_n) are of the order n and using the fact that (U_s, V_s) is a solution for $n \to 0$ we get

$$inU_n + U_0 \frac{\partial U_n}{\partial x} + U_n \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_n}{\partial y} + V_n \frac{\partial U_n}{\partial y} - \nu \frac{\partial^2 U_n}{\partial y^2} = in\left(U_0 - U_s\right)$$
$$\frac{\partial U_n}{\partial x} + \frac{\partial V_n}{\partial y} = 0$$

Subject to boundary conditions $U_n = V_n = 0$ at $y = 0$

$U_n \to 0$ as $y \to \infty$

Using a Kármán-Pohlhausen method Lighthill showed that

$$\frac{U_n}{U_0} = \frac{in\delta^2}{\nu}(1-\eta)^2\left\{A\eta + \left(2(A) - \tfrac{1}{2}\right)\eta^2\right\} \qquad \text{Eqn. 21.}$$

where $\eta = \dfrac{y}{\delta}$

and $A = f_n\left(\dfrac{\delta^2}{\nu} \dfrac{dU_0}{dx}, \dfrac{n\delta^2}{\nu} \right)$

For large values of $\dfrac{n\delta^2}{\nu}$, equations 20 simplify to:

$$inU_1 = inU_0 + \nu \frac{\partial^2 U_1}{\partial y^2}$$

where $\dfrac{\partial U_1}{\partial y} = 0$ at $y = 0$

and $U_1 \to U_0(x)$ as $y \to \infty$

thus $U_1 = U_0(x)\left(1 - e^{-y\sqrt{\frac{in}{\nu}}}\right)$ Eqn. 22.

This is equivalent to Stokes' oscillating plane solution and shows that the boundary layer only has a non instantaneous response close to the wall i.e. within a normal Stokes' layer which is of thickness $\sqrt{\nu/n}$

The skin frictions derived from equations 21 and 22 both show a phase lead over the free stream oscillating flow.

3.8. The flow near an oscillating stagnation point.

This forms a combination of two classical flows - namely plane stagnation point flow against a plane wall and an infinite wall oscillating in an otherwise stationary fluid,

The component of velocity parallel to the wall is denoted by $u = u_0 + \epsilon u_1 e^{int}$ whilst the normal component is $w = w_0 + \epsilon w_1 e^{int}$ where u_0, u_1, w_0, w_1 are functions of x and y. As y tends to infinity u_0 tends to ax and w_0 tends to $-ay + 0.65\sqrt{a\nu}$ where a = constant. Defining $\eta = y\sqrt{\frac{a}{\nu}}$ the steady velocities within the boundary layer are given by $u_0 = axf'(\eta)$ and $w_0 = -\sqrt{a\nu}f(\eta)$

The above equations for the steady velocities were obtained by Blasius when considering plane stagnation flow at the front of a large cylinder.

The problem then is to find u_1 and w_1 subject to the conditions that u_1 reduces to a given value at the oscillating wall and to zero at large distances from the wall whilst w_1 is zero at the wall. An exact solution of the Navier Stokes equations exists for all values of ϵ and n in the form $u_1 = \zeta(\eta)$, $w_1 = 0$ when ζ satisfies,

$$\zeta'' + f\zeta' - f'\zeta - \frac{in}{a}\zeta = 0$$

with
$$\zeta = 1 \quad \text{at } \eta = 0$$
$$\zeta \to 0 \quad \text{at } \eta \to \infty$$

The quasisteady solution as $\frac{n}{a}$ tends to zero is $\zeta_0 = f''(\eta)/f''(0)$.

Thus the velocity for the flow impinging on a wall moving with constant velocity ϵ is given by

$$u = axf'(\eta) + \epsilon f''(\eta)/f''(0)$$
$$w = -\sqrt{a\nu}\, f(\eta)$$

For large values of $\frac{n}{a}$ the solution contains as a first term the solution to an oscillating wall with zero mean flow mainly because $\sqrt{\nu/a}$ is much greater than $\sqrt{\nu/n}$ so that there is little interaction between the two parts of the flow.

The skin friction phase lead tends to $\pi/4$ as n/a increases and also the amplitude tends to grow as $\sqrt{\frac{n}{a}}$

3.9. Separation in an unsteady laminar boundary layer.

Separation in a laminar boundary layer is caused by the inability of the layer to remain attached when an adverse pressure gradient of sufficient value exists. This is characterised by the loss of shear stress or zero slope of the velocity profile at the wall. The fluid particles nearer the wall possess lower kinetic energy and thus when a constant negative pressure gradient is impressed on the boundary layer, the profile progressively suffers zero or negative slope from the wall outwards as the pressure gradient increases. This definition is unambiguous and generally applicable in steady flows.

However, in unsteady flow this simple criterion cannot be unequivicably applied because a zero or negative profile can occur at several points in the layer and yet not necessarily result in a boundary layer edge streamline deformation caused by viscous-inviscid interaction. The unsteady boundary layer equations predict profile inflections and zero slope points even when the boundary layer approximations are assumed valid. No inviscid interaction is predicted and thus the steady state definition of separation is not acceptable.

Despard and Miller (1971) carried out an experimental investigation of the unsteady separation of a laminar boundary layer subject to a periodically varying negative pressure gradient and free stream velocity. The results indicated that separation of such a layer, resulting in viscous - inviscid interaction and thus boundary layer outer streamline deformation could be reasonably defined as occuring when the profile gradient was zero or less over the complete cycle. They caused good collapse of data by an empirical formula given below;

$$\Delta_s = \frac{x_{ss} - x_s}{x_{ss}} = 1.13 \times 10^7 \, R_e^{-1.77} \, P^{-0.28} \, f^{-0.26} \, A^{0.033}$$

where

$$P = \frac{\int_{x_0}^{x_s} x \frac{d\bar{p}}{dy} \, dx}{\int_0^{x_s} x \left(\frac{d\bar{p}}{dx}\right) dx \left(\frac{dp}{dx}\right)_{x_s} \frac{x_0}{q}}$$

where Δ_s = fraction movement of separation point.

x_{ss} = steady flow separation point.

x_s = unsteady flow separation point (Note that by the definition given above, this point is stationary)

x_0 = point at which the pressure gradient becomes positive for the last time.

$Re = \frac{U x_s}{\nu}$

$f = \frac{n \nu}{U^2}$

$A = \Delta U / U$

q = dynamic head

U = free stream mean velocity

ΔU = free stream velocity perturbation.

n = harmonic frequency of flow oscillation.

The strong dependance of separation point movement on Reynolds number is obvious since the faster flow boundary layers have greater kinetic energy and thus for the same flow perturbation are less affected than their slower flow counterparts. The frequency dependance is such that as the frequency rises, the unsteady separation point moves towards the steady flow point. This is also intuitive because as the frequency rises the Stokes layer becomes non interactive with the main layer by virtue of its thinness $\left(0\sqrt{\frac{\nu}{n}}\right)$ and the main layer merely responds quasisteadily to the flow velocity changes. Thus at high frequencies the dominant viscous effects at the wall obviate separation.

4.1. A Turbulent Boundary layer with a fluctuating free stream.

The major difficulty in a mathematical treatment of this problem is that the responses due to a series of small changes of input cannot be superimposed to give a final response as in linear problems. The unsteady turbulent boundary layer depends on the instantaneous free stream velocity and also its past history.

Karlsson (1958) conducted an experimental investigation into this problem and found that for frequencies from 0-48 Hz and free stream fluctuations of up to 34% of the mean, barely detectable non instantaneous response occured.

He concluded that at a Reynolds number based on momentum thickness of 3.6×10^3, the instantaneous boundary layer could be calculated from a knowledge of the instantaneous free stream velocity. The fluctuating free stream had very little fluctuating static pressure.

It is intuitive that the response of a boundary layer to a fluctuating pressure gradient be non linear owing to viscous inviscid interaction. It is possible that the outer edge of the turbulent boundary layer responds in such a way and also that the fluid adjacent to the surface changes character from viscous to inviscid flow if separation is imminent. These likely properties of boundary layers subjected to fluctuating pressure gradients seem a probable explanation of the effect of stall hysterisis on aerofoils. Goldstein contains an account of this phenomenum and it is thus known from experiment that a fast incidence increase near stall can result in a high lift increment - greater than that expected from quasi-steady theory. The delay of separation seems the probable cause. Conversely, a sharp drop in incidence results in an "undershoot" of lift probably caused by non reattachment of the flow.

5.1. Concluding Remarks.

The principle features of incompressible unsteady boundary layer analysis have been outlined and it is evident from the works described that the most important properties were discovered by Stokes and Rayleigh prior to Prandtl's boundary layer theory.

Stokes' first problem, the accelerated wall evolves and becomes the Blasius steady state laminar boundary layer, whilst his second problem, the oscillating wall provides the inner wall solution to the work done by Lin and Lighthill. Also, the well known parameter diffusion time results from the analysis. (Diffusion time = δ^2/ν where δ = distance over which the vorticity or momentum transport acts and ν the kinetic viscosity.).

Rayleigh discovered mathematically the existance of steady streaming or an outer boundary layer near an oscillating body. This results from net mean motion caused by the

Reynolds' stresses generated by the oscillating flow and thus explains the motion of dust particles in a Kundt tube.

Finally, little theoretical work has been done on turbulent boundary layers because its non linearity obviates exact or simple approximated solutions.

However, by assuming that the effective viscosity is much higher than ν , as in steady state mixing length theories, one would expect that the Stokes' layer would be much thicker and thus be a proportionately more important part of an unsteady turbulent boundary layer. The experimental work started by Karlsson did not verify this because the laminar sublayer was almost three times as deep as the Stokes' layer and thus a turbulent Stokes' layer could not occur.

It was finally pointed out that probably the most important unsteady boundary layer effect resulted from separation point motion thus allowing viscous - inviscid interaction.

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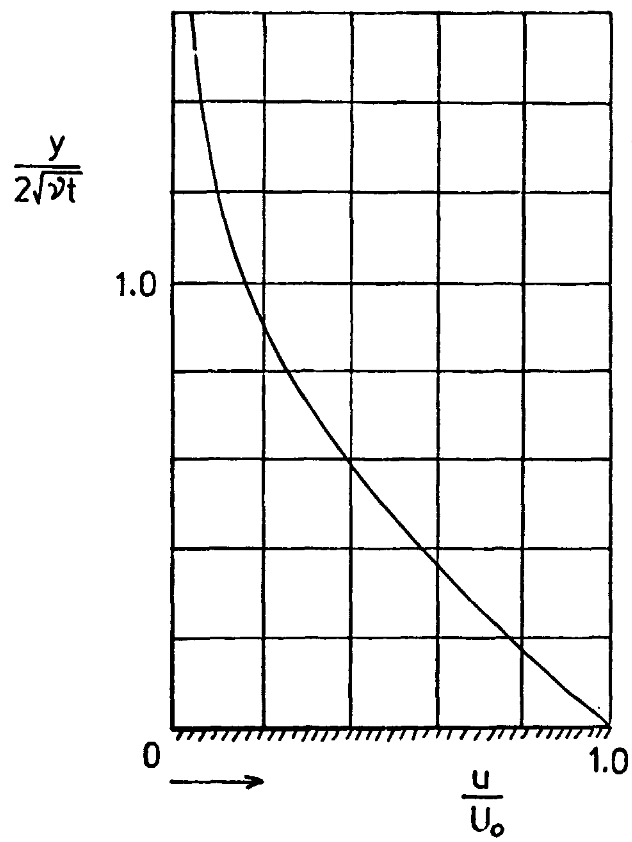

FIG. 1. Velocity distribution above a suddenly accelerated wall

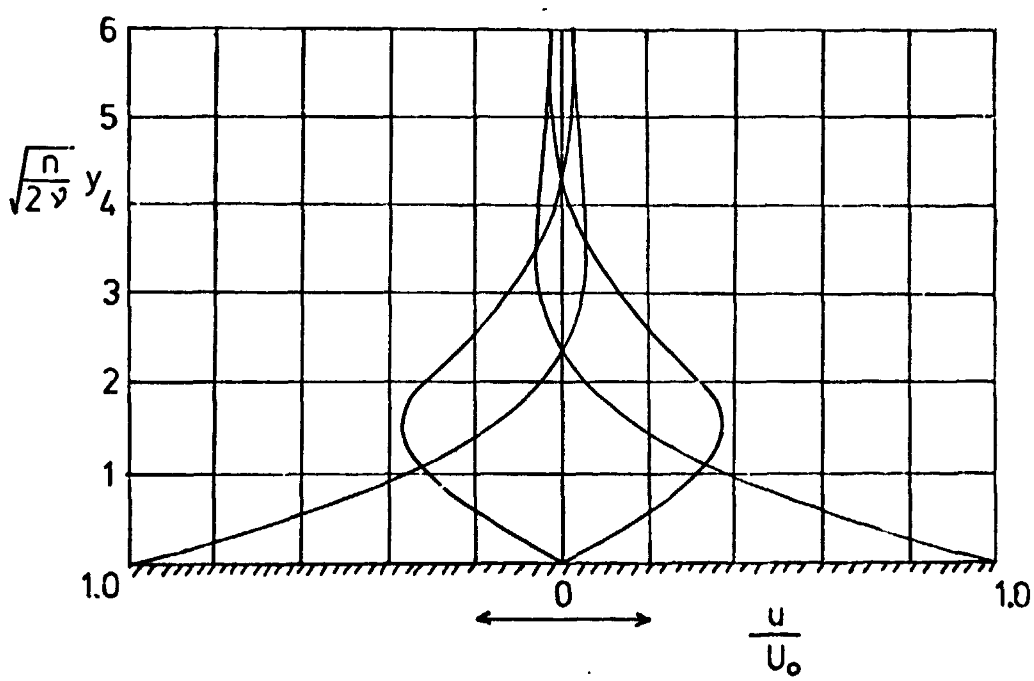

FIG. 2. Velocity distribution near an oscillating wall

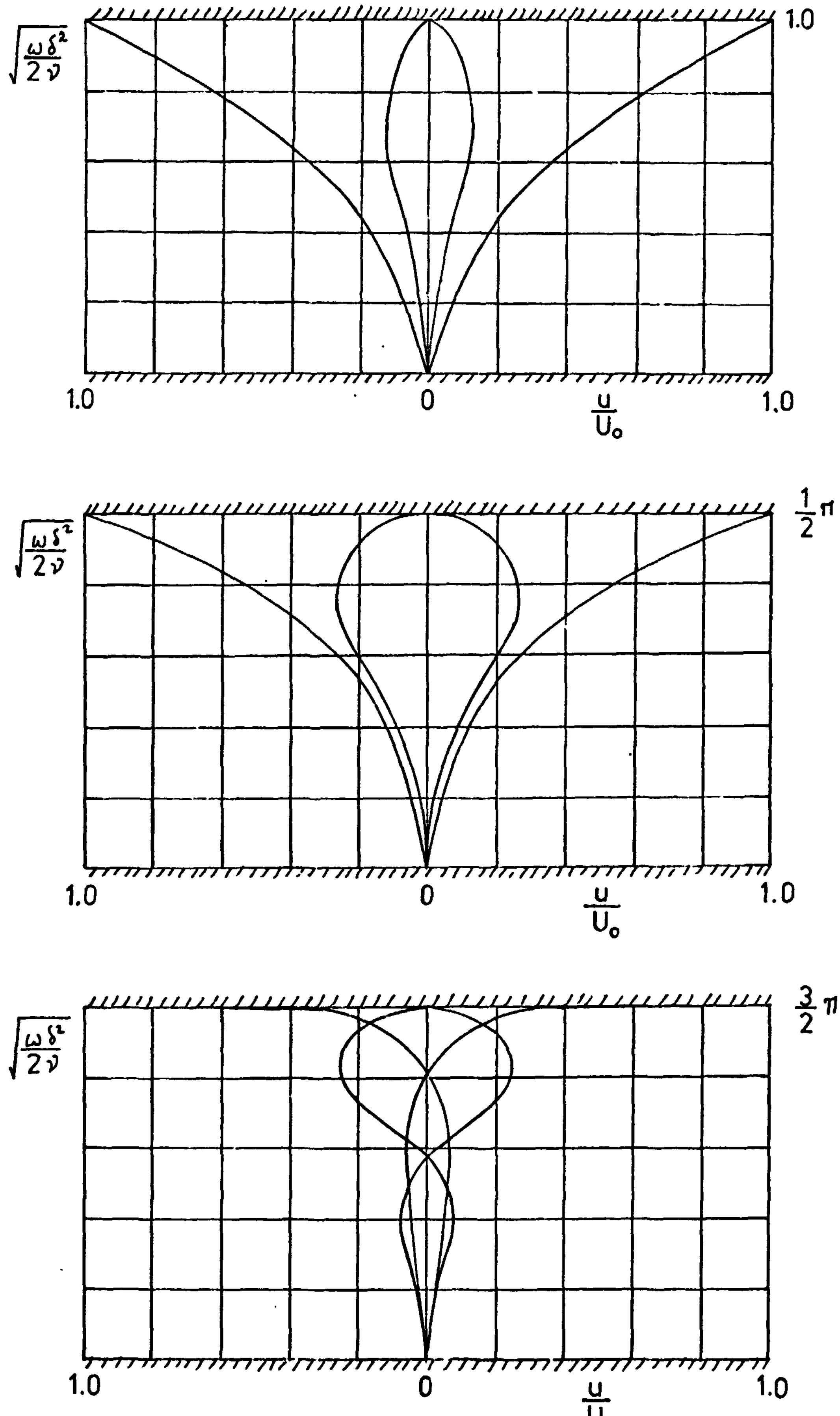

FIG. 3. Velocity distribution between oscillating walls for various values of $\sqrt{\frac{\omega \delta^2}{2\nu}}$